

Juraj PAULECH\*, Justín MURÍN\*\*, Juraj HRABOVSKÝ\*\*\*, Vladimír KUTIŠ\*\*\*\*

## NEW LINK FINITE ELEMENT FOR ELECTRO-THERMAL ANALYSES OF FGM MATERIALS

### NOVÝ PRÚTOVÝ KONEČNÝ PRVOK PRE ELEKTRO-TEPELNÉ ANALÝZY FGM MATERIÁLOV

#### Abstract

In this contribution, new electro-thermal link finite element for Functionally Graded Materials (FGM) will be presented. Spatial variation of material properties as well as varying convection effect will be considered. FEM equations describing the behaviour of the link element will be based on semi-analytical calculation of ordinary differential equations describing coupled electro-thermal problem.

#### Abstrakt

V predkladanom príspevku bude prezentovaný nový elektro-tepelný prúťový konečný prvok pre Funkcionálne Gradované Materiály (FGM), pričom bude uvažovaná priestorová premenlivosť materiálových vlastností a pôsobenie premenlivej konvekcie na prút. Budú prezentované nové MKP rovnice opisujúce správanie sa prúťu, založené na semi-analytickom výpočte obyčajných diferenciálnych rovníc popisujúcich previazanú elektro-tepelnú úlohu.

#### Keywords

New link element, FGM, Coupled analysis, Joule heat, FEM equations

## 1 INTRODUCTION

Nowadays, new materials are necessary for sophisticated structures like MEMS systems, advanced electronic devices, etc. Computer modelling of such complex systems, like structures with spatial variation of material properties (e.g. FGM), using commercial FEM code with classic elements, needs remarkable effort during preparation phase and sufficient computer equipment for solution phase because of necessity the numbers of elements and material models.

This contribution deals with derivation of new 1D link finite element for one-way coupled static electro-thermal analyses of FGM system considering Joule heat and also non-constant convective heat transfer.

---

\* Ing., Department of Applied Mechanics and Mechatronics, IPAAE, Faculty of Electrical Engineering and Information Engineering (FEI) STU Bratislava, Ilkovicova 3, Bratislava, Slovakia, tel. (+421) 2 60291452, e-mail: juraj.paulech@stuba.sk

\*\* prof., Ing., DrSc., Department of Applied Mechanics and Mechatronics, IPAAE, FEI STU Bratislava, Ilkovicova 3, Bratislava, Slovakia, tel. (+421) 2 60291611, e-mail: justin.murin@stuba.sk

\*\*\* Ing., Department of Applied Mechanics and Mechatronics, IPAAE, FEI STU Bratislava, Ilkovicova 3, Bratislava, Slovakia, tel. (+421) 2 60291559, e-mail: juraj.hrabovsky@stuba.sk

\*\*\*\* doc., Ing., PhD., Department of Applied Mechanics and Mechatronics, IPAAE, FEI STU Bratislava, Ilkovicova 3, Bratislava, Slovakia, tel. (+421) 2 60291562, e-mail: vladimir.kutis@stuba.sk

## 2 DERIVATION OF NEW FEM EQUATIONS

Derivation process of the new FEM equations for coupled electro-thermal link element is based on differential equations for electric and thermal fields for 1D type of analysis, respectively:

$$\begin{aligned} -\sigma(x) \frac{d^2 \varphi(x)}{dx^2} - \frac{d\sigma(x)}{dx} \frac{d\varphi(x)}{dx} &= 0 \\ -\lambda(x) \frac{d^2 T(x)}{dx^2} - \frac{d\lambda(x)}{dx} \frac{dT(x)}{dx} + \alpha(x) T(x) \frac{o}{A} &= Q(x) + \alpha(x) T_{amb} \frac{o}{A} \end{aligned} \quad (1)$$

where  $\sigma(x)$  [S/m] is variable electric conductivity,  $\varphi(x)$  [V] is distribution of the electric potential along the link conductor,  $\lambda(x)$  [W/mK] is non-constant thermal conductivity,  $T(x)$  [K] is absolute temperature along the conductor,  $\alpha(x)$  [W/m<sup>2</sup>K] is non-constant coefficient of convective heat transfer,  $o$  [m] is perimeter of the link conductor,  $A$  [m<sup>2</sup>] is conductor's cross-section,  $Q(x)$  [W/m<sup>3</sup>] is volume heat source in the conductor and  $T_{amb}$  [K] is ambient temperature.

Let us consider the FGM link conductor according to Fig. 1. The solution of these differential equations is based on numerical method for solving 1D differential equation with non-constant coefficients and with right-hand side described in [1] in detail. This method assumes polynomial form for the coefficients on the left-hand side and right-hand side itself. Solutions for electric and thermal fields have then, according to [1], the general form:

$$\begin{aligned} \varphi(x) &= c_{0e}(x)\varphi_0 + c_{1e}(x)\varphi'_0 \\ T(x) &= c_{0t}(x)T_0 + c_{1t}(x)T'_0 + \sum_{j=0}^g \varepsilon_j b_{j+2}(x) \end{aligned} \quad (2)$$

where  $c_{0e}(x)$  and  $c_{1e}(x)$  are so-called transfer functions for uniform solution of the electric differential equation,  $\varphi_0$  and  $\varphi'_0$  are electric potential and the first derivation of electric potential for the beginning of the conductor (node 0, see Fig. 1), respectively. Next,  $c_{0t}(x)$  and  $c_{1t}(x)$  are transfer functions for uniform solution and  $b_{j+2}(x)$  is transfer function for of the particular solution of the thermal differential equation. Again,  $T_0$  and  $T'_0$  are temperature and the first derivation of temperature for the beginning of the conductor, respectively. The degree of the right-hand side polynomial is  $g$  [-] and coefficients  $\varepsilon_j$  are constant coefficients of the right-hand side.

It is necessary to remark that used method for solution of 1D differential equations allows calculation only for chosen point  $x$  of the considered interval (length of the link). So for calculation of any physical field in the conductor it is necessary to calculate it for chosen number of individual points – positions  $x$ .

The final 1D polynomial change of the real FGM conductor's electric and material properties ( $\sigma$  and  $\lambda$ ) can be calculated according to homogenization process further described in [2, 3].

One-way coupling between the electric and thermal field is provided by Joule heat  $P_j(x)$  [W/m<sup>3</sup>], that can be calculated as one of the outputs from electric analysis and it enters the thermal analysis as volume heat (beside or instead of  $Q(x)$ ). We can calculate the volume Joule heat as:

$$P_j(x) = \frac{J^2(x)}{\sigma(x)} \quad (3)$$

As we can see from (3),  $P_j(x)$  has not the polynomial form, so it is necessary to transform it to the polynomial. Our program script automatically finds the most appropriate polynomial  $\hat{P}_j(x)$  for the volume Joule heat distribution. The procedure is based on rising degree polynomial interpolation through the chosen points of  $P_1(x)$ , accuracy is given by comparison its definite integral with those calculated using non-polynomial form.

Considering constitutive laws for electric and thermal fields:

$$\begin{aligned} J(x) &= -\sigma(x)\varphi'(x) \\ q(x) &= -\lambda(x)T'(x) \end{aligned} \quad (4)$$

where  $J(x)$  [A/m<sup>2</sup>] is electric current density and  $q(x)$  [W/m<sup>2</sup>] is thermal heat flux in the conductor, and using chosen boundary conditions for electric and thermal field, we can derive new FEM equations for these fields (lower indexes “0” and “L” mean that value of the physical quantity is expressed for node 0 or node L, respectively):

$$\begin{bmatrix} c_{0e}(L) & -1 \\ -\left(c_{0e}(L) - \frac{c_{1e}(L)c'_{0e}(L)}{c'_{1e}(L)}\right) & 1 \end{bmatrix} \begin{bmatrix} \varphi_0 \\ \varphi_L \end{bmatrix} = \begin{bmatrix} \frac{c_{1e}(L)}{\sigma_0} J_0 \\ \frac{c_{1e}(L)}{c'_{1e}(L)\sigma_L} J_L \end{bmatrix}$$

$$\begin{bmatrix} c_{0t}(L) & -1 \\ -\left(c_{0t}(L) - \frac{c_{1t}(L)c'_{0t}(L)}{c'_{1t}(L)}\right) & 1 \end{bmatrix} \begin{bmatrix} T_0 \\ T_L \end{bmatrix} = \begin{bmatrix} \frac{c_{1t}(L)}{\lambda_0} q_0 - \sum_{j=0}^g \varepsilon_j b_{j+2}(L) \\ \frac{c_{1t}(L)}{c'_{1t}(L)\lambda_L} q_L - \frac{c_{1t}(L)}{c'_{1t}(L)} \sum_{j=0}^g \varepsilon_j b'_{j+2}(L) + \sum_{j=0}^g \varepsilon_j b_{j+2}(L) \end{bmatrix} \quad (5)$$

Using same assumptions it is also possible to derive the expressions for electric potential and temperature also within the conductor (at chosen position  $x$ ):

$$\varphi(x) = \left[ c_{0e}(x) - c_{0e}(L) \frac{c_{1e}(x)}{c_{1e}(L)} \right] \varphi_0 + \left[ \frac{c_{1e}(x)}{c_{1e}(L)} \right] \varphi_L$$

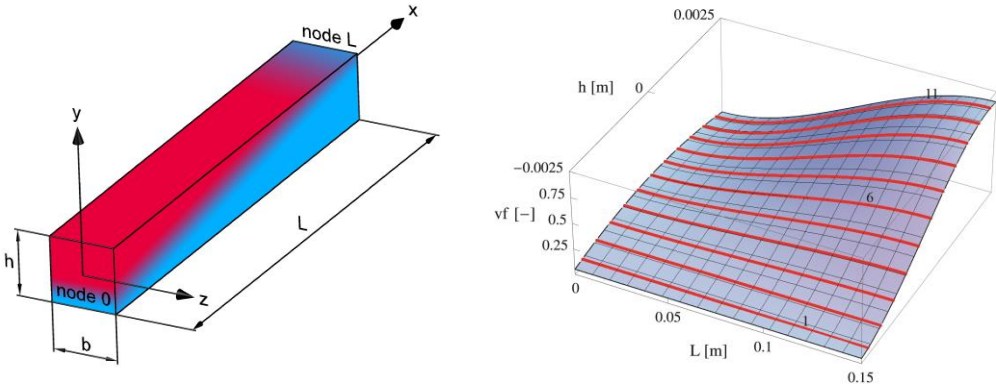
$$T(x) = \left[ c_{0t}(x) - c_{0t}(L) \frac{c_{1t}(x)}{c_{1t}(L)} \right] T_0 + \left[ \frac{c_{1t}(x)}{c_{1t}(L)} \right] T_L - \frac{c_{1t}(x)}{c_{1t}(L)} \sum_{j=0}^g \varepsilon_j b_{j+2}(L) + \sum_{j=0}^g \varepsilon_j b_{j+2}(x) \quad (6)$$

### 3 NUMERICAL EXPERIMENT

In this chapter one academic example of calculation of one-way coupled electro-thermal field in given FGM link conductor will be presented. The task will be solved using our new approach and also by commercial FEM code ANSYS [4] and by numerical solver for differential equations in software Mathematica [5] due to comparison reasons.

Let us consider a link conductor with square cross-section. Its length is  $L = 150$  [mm], height and depth are  $h = b = 5$  [mm], see Fig. 1. Let the conductor consists of mixture of two component materials – matrix (index  $m$ ) with constant electric conductivity  $\sigma_m(x, y) = 1.429 \times 10^6$  [Sm<sup>-1</sup>] and thermal conductivity  $\lambda_m(x, y) = 5$  [Wm<sup>-1</sup>K<sup>-1</sup>], and fibre (index  $f$ ) with electric conductivity  $\sigma_f(x, y) = 1.111 \times 10^7$  [Sm<sup>-1</sup>] and thermal conductivity  $\lambda_m(x, y) = 58$  [Wm<sup>-1</sup>K<sup>-1</sup>]. Volume fraction of individual components is functionally changed according to chosen polynomial.

Fibre volume fraction with representation of individual layers ( $N = 11$ , see [2, 3] for details) used for calculation of homogenized material properties is shown in Fig. 1.



**Fig. 1** FGM link conductor (left) and change of the fibre volume fraction in the conductor (right)

We assume static state for electro-thermal analysis considering also non-constant convection from the conductor surface with polynomial convective coefficient  $\alpha(x)$  and constant ambient temperature  $t_{amb}$ . There is the electric potential  $\varphi(L)$ , electric current  $I(0)$  (entering the node), Celsius temperature  $t(0)$  and the heat flux  $q(L)$  (entering the node) specified for the nodes 0 and L, respectively. Boundary conditions, see Fig. 2, are then:

$$I(0) = 50 \text{ A} \rightarrow J(0) = 2 \times 10^6 \text{ [Am}^{-2}\text{]}$$

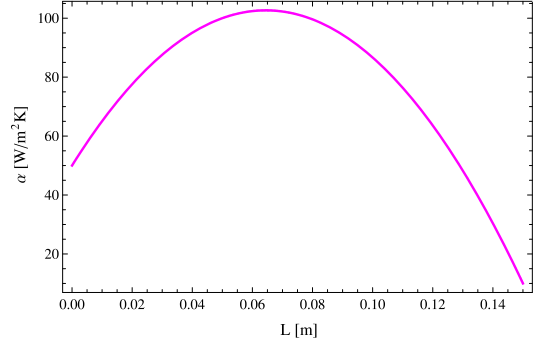
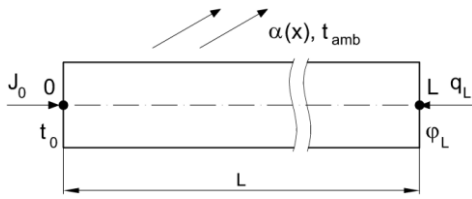
$$\varphi(L) = 1 \text{ [V]}$$

$$t(0) = 25 \text{ [}^\circ\text{C]}$$

$$q(L) = 12\,000 \text{ [Wm}^{-2}\text{]}$$

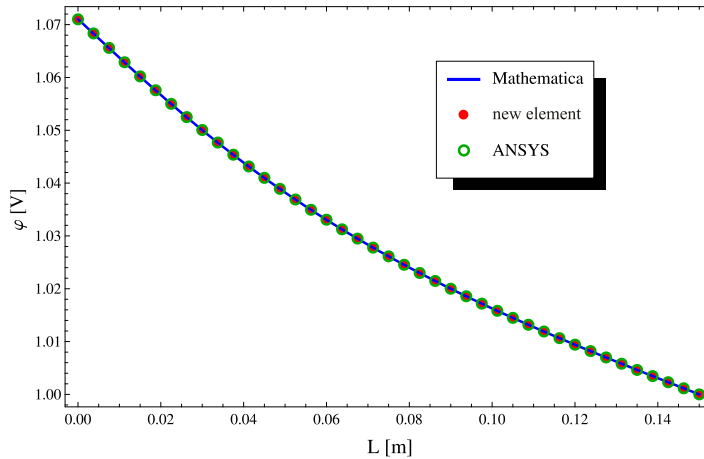
$$\alpha(x) = 50 + \frac{4\,900}{3}x - \frac{38\,000}{3}x^2 \text{ [Wm}^{-2}\text{K}^{-1}\text{]}$$

$$t_{amb} = 25 \text{ [}^\circ\text{C]}$$

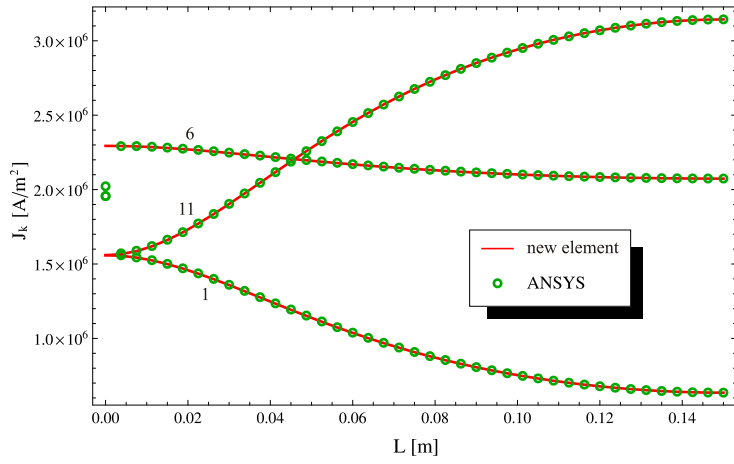


**Fig. 2** Boundary conditions of the model

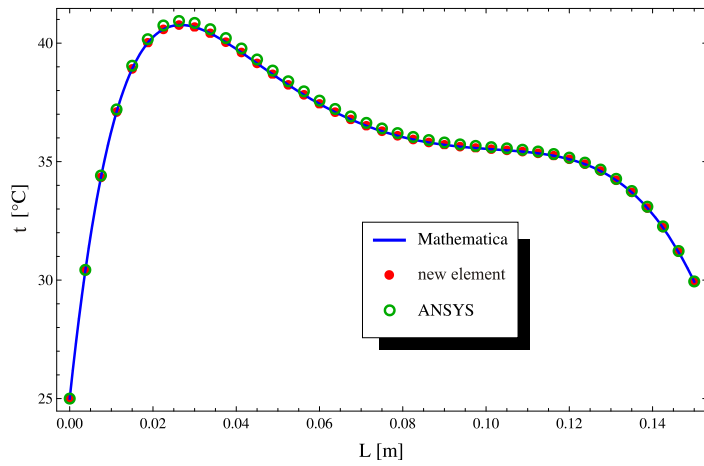
We also created 3D model in code ANSYS where we used more than 560 000 SOLID69 elements (8-node bricks). We considered only a half of the model (symmetry in the depth direction). The task was also solved in software Mathematica, where the differential equations (1) with homogenized electric and thermal material properties and specified boundary conditions were numerically solved. Finally, the task was also solved by one our new developed two-nodal link element using FEM equation (5) for nodal points of the link and using equation (6) for chosen points in the field of the link (41 points together). In Fig. 3 we can see calculated longitudinal electric potential distribution in the link and in Fig. 4 there is distribution of the current densities for chosen layers (1<sup>st</sup>, 6<sup>th</sup> and 11<sup>th</sup>). In Fig. 5 we can see calculated longitudinal temperature distribution in the link and in Fig. 6 there is distribution of the heat fluxes for chosen layers (1<sup>st</sup>, 6<sup>th</sup> and 11<sup>th</sup>). Summary of calculated results is in Tab. 1.



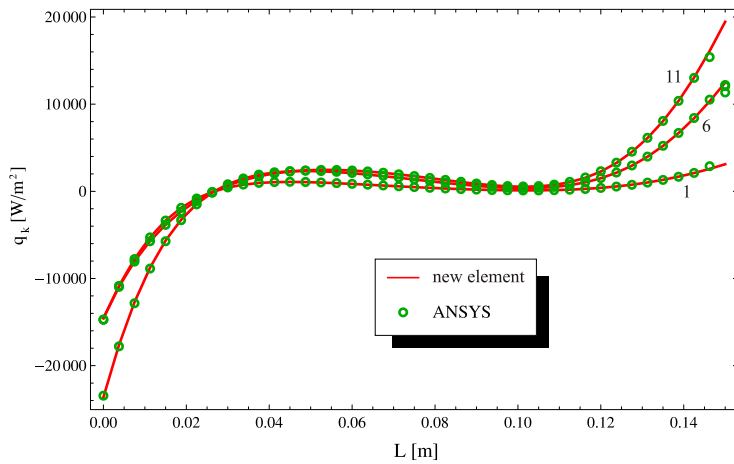
**Fig. 3** Distribution of the electric potential through the length of conductor



**Fig. 4** Longitudinal distribution of the current densities in the chosen layers of conductor



**Fig. 5** Temperature distribution through the length of the link



**Fig. 6** Longitudinal distribution of the heat fluxes in chosen layers of the link

**Tab. 1** Comparison of calculated electric and thermal quantities in nodal points of the link system

	$\varphi_0$ [V]	$J_{1,L}$ [Am <sup>-2</sup> ]	$J_{6,L}$ [Am <sup>-2</sup> ]	$J_{11,L}$ [Am <sup>-2</sup> ]	$J_L^H$ [Am <sup>-2</sup> ]
<b>new element</b>	1.07104	$6.35288 \times 10^5$	$2.07371 \times 10^6$	$3.14359 \times 10^6$	$2 \times 10^6$
<b>ANSYS</b>	1.07100	$6.35636 \times 10^5$	$2.07435 \times 10^6$	$3.14410 \times 10^6$	-
<b>Mathematica</b>	1.07104	-	-	-	$2 \times 10^6$
	$t_L$ [°C]	$q_{1,0}$ [Wm <sup>-2</sup> ]	$q_{6,0}$ [Wm <sup>-2</sup> ]	$q_{11,0}$ [Wm <sup>-2</sup> ]	$q_0^H$ [Wm <sup>-2</sup> ]
<b>new element</b>	29.9335	-14 556.0	-23 612.7	-14 556.0	-19 990.0
<b>ANSYS</b>	29.9436	-14 706.1	-23 444.9	-14 735.6	-
<b>Mathematica</b>	29.9331	-	-	-	-19 990.7

We can see that obtained results correspond to ANSYS 3D simulation very well. But there is notable difference for current densities near the node 0 and for heat fluxes results near the node L. It is caused due to the fact that 3D ANSYS model has to fulfil boundary conditions  $J_0$  and  $q_L$  for every element in these nodal areas. This behaviour and difference in results suggests to the local effect caused by simplification of the real system into a 1D system. Tab. 1 contains also values for homogenized current density and heat flux (superscript  $H$ ) for homogenized link element.

#### 4 CONCLUSION

New finite link element for one-way coupled static electro-thermal analyses has been developed in this contribution. New FEM equations with consideration Joule heat and non-constant convective heat transfer were derived. Numerical examples with good agreement between calculations with just only one new link element and commercial FEM code that uses numbers of classic elements have been presented. The new approach fully agrees with numerical solution for 1D differential equation of the electric and thermal fields. So, effectiveness and accuracy of the new developed link element for these analyses are excellent.

#### ACKNOWLEDGEMENT

This contribution is the result of the project implementation Centre of Competence in New Materials, Advanced Technologies and Energy, ITMS 26240220073 supported by the Research & Development Operational Programme funded by the ERDF.

This work was supported by Grant Agency VEGA, project No. 1/0534/12.

This work was supported by Grant Agency KEGA, project No. 015STU-4/2012.

#### REFERENCES

- [1] RUBIN, H.: Analytische Lösung linearer Differentialgleichungen mit veränderlichen Koeffizienten und baustatische Anwendung. *Bautechnik* 76. 1999.
- [2] ALTENBACH, H. et al.: Mechanics of composite structural elements. *Springer-Verlag*. Berlin. 2003, 468 p.
- [3] MURÍN, J., KUTIŠ, V.: Improved mixture rules for the composite (FGM's) sandwich beam finite element. In: *Computational Plasticity IX. Fundamentals and Applications*. Barcelona. (2007), p. 647-650.
- [4] ANSYS Swanson Analysis System, Inc., 201 Johnson Road, Houston, PA 15342/1300, USA.
- [5] S. Wolfram MATHEMATICA 5, Wolfram research, Inc., 2003.